The Harmonic Series

A path to understanding musical intervals, scales, tuning and timbre.

by

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When musicians use the term *overtone series*, they are generally referring to a set of frequency components that appear above a musical tone. The related term *harmonic series* is a more precisely defined concept with applications in both music and mathematics. Though musicians sometimes use these terms interchangeably, the term harmonic series specifically refers to a series of numbers related by whole-number ratios. For example, the series of frequencies 1000, 2000, 3000, 4000, 5000, 6000, etc., given in Hertz (Hz.), is a harmonic series; so is the series 500, 1000, 1500, 2000, 2500, 3000, etc. Notice that the difference in frequency between adjacent members of both series is constant, that is to say, the harmonics are equally-spaced. The lowest frequency component of such a series is called the *fundamental*. For example, the fundamental of the first series is 1000 Hz, and the fundamental of the second series is 500 Hz. The other (higher) frequency components are called overtones, harmonics or partials.

> **o•ver•tone** "One of the frequency components of a sound other than that of lowest frequency. Usually overtones are numbered consecutively in ascending order of frequency; they need not be harmonic."
>
> Murray Campbell, *The New Grove Dictionary of Music*

A compact way to express this type of series is to use *frequency-multiple notation*. For example, 1000, 2000, 3000, 4000, 5000, 6000, etc. and 500, 1000, 1500, 2000, 2500, 3000, etc., may both be expressed as:

$$f_0,\ 2\,f_0,\ 3\,f_0,\ 4\,f_0,\ 5\,f_0,\ 6\,f_0,\ \text{etc.} \tag{1}$$

where f_0 is the fundamental frequency, and the whole numbers to the left of each f_0 indicates the respective frequency multiplier.

A harmonic series may also be expressed using musical staff notation as shown in Fig. 1.

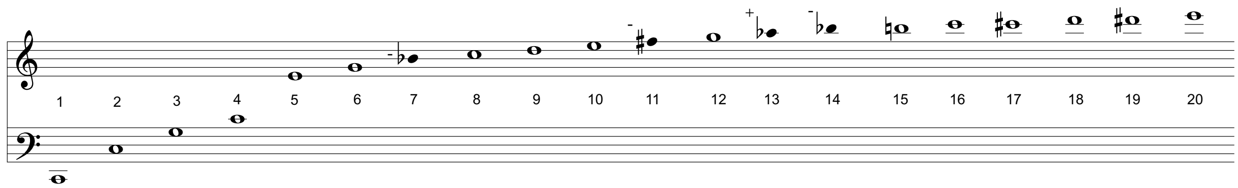

Fig. 1. The first 20 partials of a harmonic series for the fundamental pitch C2 (ca. 65.4 Hz.) expressed in traditional musical staff notation with frequency multiples indicated between the staves. The - and + symbols indicate that the notated pitch is significantly lower or higher, respectively, than the same pitch on a modern piano.

This traditional way of notating the harmonic series is obviously less precise at conveying frequency information than the frequency-multiple notation shown above in (1). However, it does provide a convenient way for musicians to memorize the harmonic series as a kind-of chord/scale of nature. Once memorized it may easily be transposed to any fundamental pitch. For example, Fig. 2 shows a similar diagram for the first 20 partials of a harmonic series on A1 (ca. 55 Hz.).

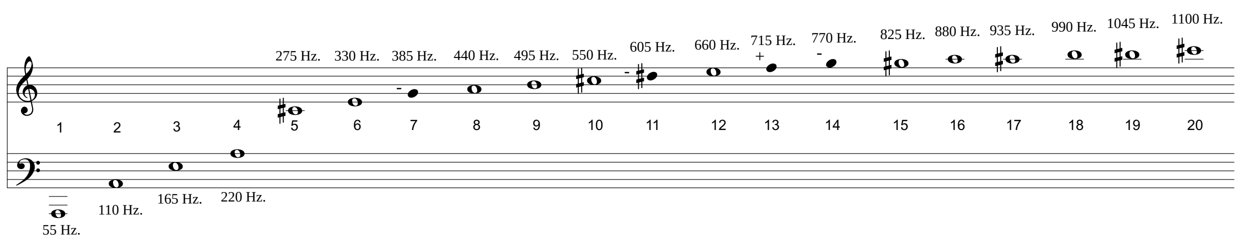

Fig. 2. The first 20 partials of a harmonic series for the fundamental pitch A1 (ca. 55 Hz.).

In Fig. 2, the frequency of each partial has been also been specified. (The frequency of each partial may be calculated by multiplying the fundamental frequency, in this case 55 Hz., by the partial number indicated between the staves.) A simple explanation for why such a pattern of harmonics (overtones) appears above the pitch we actually perceive may be found in the physical model of a plucked string. We will explore this model and the relationship between string length and music interval in the next section in order to gain a deeper understanding of why this pattern occurs in nature.

Before we move on to the next section, it should be noted that composer Paul Hindemith's book *The Craft of Musical Composition* (1942) contains one of the last century's best known introductions to the role of the harmonic series in music. This article, inspired by Hindemith's presentation (among others such as Cowell 1930), charts a similar course to the Hindemith. Interested readers looking for a more in depth discussion of these issues may wish to consider reading the Hindemith, as well as the other references listed at the end of the article.

String Length and Musical Interval

Fig. 3 shows an illustration of a *monochord* that appeared in Hindemith's book.

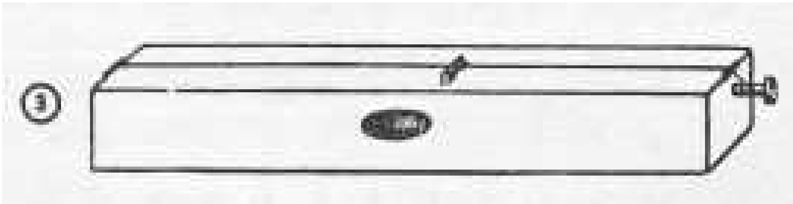

Fig. 3. A monochord.

A monochord is a simple instrument used by theorists since the Middle Ages to investigate the relationship between string length and pitch. A monochord has three main components: 1) a string fixed at both ends, 2) a moveable bridge, and 3) a resonating body. When plucked, the monochord's string vibrates at a rate directly proportional to its length. (The string's density and tension must also be taken into account, but for the sake of simplicity let's leave those variables aside for now.) However, it has been demonstrated experimentally that this is not the only mode of vibration of the string. In addition to vibrating over its entire length, a string simultaneously vibrates over fractional divisions of its length (1/2, 1/3, 1/4, 1/5, 1/6, etc.; See also, Fig. 4) producing a series of harmonics (overtones) whose frequencies are inversely proportional (2*x*, 3*x*, 4*x*, 5*x*, 6*x*, etc., where *x* is the fundamental frequency of the string) to those fractional divisions. Theoretically an infinite number of these multiple modes of vibration exist, each mode producing its own harmonic. As one ascends the series, the amplitude, or loudness of each harmonic tends to diminish, so the higher modes produce harmonics that are usually too soft for us to hear.

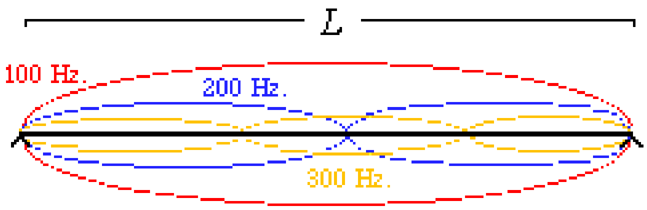

Fig. 4. The first three simultaneous modes of vibration of a string whose length is *L* (not to scale).

Research by music psychologists has shown that the ear/brain system tends to fuse harmonically-related frequency components into a single sensation we call *pitch*. So that rather than perceiving the many individual harmonics of a musical tone, we ordinarily perceive an identifiable tone color, or timbre, whose pitch is associated with the fundamental of the harmonic series being experienced.

> "…pitch is the intersubjective correlate of frequency."
>
> Robert Morris, *Composition with Pitch-Classes*

It should be noted that a series of frequency components that are nearly harmonically-related, such as,

$$f_0,\ 2.01\, f_0,\ 3.05\, f_0,\ 3.99\, f_0,\ 4.98\, f_0,\ 6.02\, f_0, \text{ etc.}$$

also produces a sensation of pitch. The process by which the brain tends to fuse individual *pure tone*, or sinusoidal, components together to form a single pitch sensation is called *fusion*. A stunning fact uncovered by music psychologists doing research on fusion in the 1920's was that the omission of the fundamental frequency from a series of harmonically-related components does not change our sensation of pitch. Fusion is but one example of the link that exists between the harmonic series and our perception the timbre of a particular instrument. We will explore the issue of timbre further near the end of this article.

The Path to Just Intervals

The discovery of a numerical relationship between string length and musical interval is commonly attributed to the Greek philosopher and theorist Pythagoras (*c*. 550 BCE). Returning to the plucked string model discussed in the previous section, we say that the pitch produced by a string of length $L/2$ sounds an octave higher than the pitch produced by a string of length L. (Again, please keep in mind that for two different strings, both strings must have the same density and tension for this to be the case.) This relationship between string length and musical interval provides musicians with a precise way to express the size of a musical interval. For example, let's divide a string into two equal parts, a *string length division ratio* of 1:2. When we pluck each 1:2 division, or string segment, a pitch related by an *octave* (i.e., the interval whose frequency is 2:1) will sound. Say we divide a string into three equal parts and pluck each of those segments. Each segment will produce a pitch three times that of the original or a perfect twelfth higher (i.e., the interval whose frequency ratio is 3:1). Fig. 5 shows another diagram from Hindemith's *The Craft of My Musical Composition*. It was designed to illustrate three basic string divisions (1:1, 1:2 and 1:3) and the pitches that will be produced when each segment is plucked.

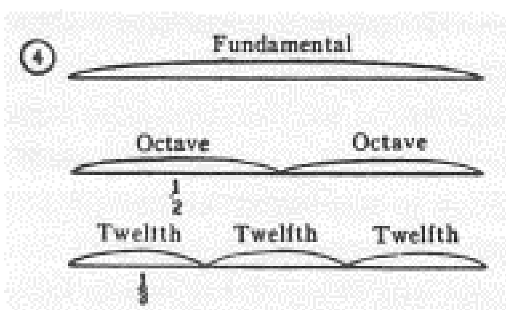

Fig. 5. String length division diagram from Hindemith's *The Craft of My Musical Composition*.

String length ratios and *interval frequency ratios* are usually expressed in their simplest form, and may alternatively be written using a division symbol '/' in place of the ratio ':' symbol. For example, the interval whose frequency ratio is 6:4, would normally be simplified to 3:2, which may also be written 3/2. Finally, it should be stated that string lengths and interval frequency ratios exhibit a *reciprocal* relationship, that is, the top and bottom numbers involved in the fractions switch places.

Because of their association with the intervals found in a harmonic series, intervals such as 2:1, 3:1, 3:2, 4:3, 5:3 are often called *natural intervals.* What is more, Fig. 1 can be used to look up which natural interval frequency ratio corresponds to which traditional tonal interval name (major third, minor third, perfect fifth, etc.), providing a convenient reference for musicians. For example, let's look up the natural interval frequency ratio for a perfect fifth. To do this, search Fig. 1 from left-to-right, looking for the first perfect fifth you encounter between any two members of the series. Notice that a perfect fifth is formed by the second and third members of the series. Consequently, the natural interval frequency ratio for the perfect fifth can be expressed as 3:2. Confirm this finding by searching for another perfect fifth higher up the series. Verify that the perfect fifth you find can be simplified to the ratio 3:2. The frequency-multiple numbers between the staves in Fig. 1 can be used to determine the interval frequency ratio for any natural interval. Since the term natural interval frequency ratio is quite a mouthful, composers and theorists more commonly refer to such intervals as *just* intervals. To gain a deeper understanding of the correspondence between traditional tonal interval names and just interval ratios, use Fig. 1 now to look up the traditional tonal intervals listed in Fig. 6a.

a. perfect fifth, b. perfect fourth, c. major sixth, d. major third, e. minor third
f. minor sixth, g. minor seventh, h. major second, i. major seventh, j. minor second

Fig. 6a. Common traditional tonal interval names.

You will find these intervals to be equivalent to the following just ratios, respectively:

a. 3:2, b. 4:3, c. 5:3, d. 5:4, e. 6:5
f. 8:5, g. 9:5, h. 9:8, i. 15:8, j. 16:15

Fig. 6b. Just ratios corresponding to the
traditional tonal interval names shown in Fig. 6a

Notice how all of the numbers involved in the ratios shown in Fig. 6b are either a multiple of 2, or a multiple of a prime number not greater than 5. As such, these ratios are said to be members of a *just 5-limit* tuning system. In his book *Genesis of a Music* (1949), composer Harry Partch assigned names to these and many other just intervals found in the lower reaches of the harmonic series. Click here to go to a page where you can hear these intervals, as well as learn the names Partch assigned them. As you can see, it is definitely worth one's time to memorize the harmonic series on a fundamental of C2 up to the sixteenth partial as shown in Fig. 1. Once memorized on C2, this pattern may easily be transposed to any other fundamental pitch.

On the horns of dilemma
"On the whole of the historical period of instrumental music, Western music has based itself upon an acoutical lie. In our time this lie--that the normal musical ear hears twelve equal intervals within the span of an octave--has led to the impoverishment of pitch usage in our music."
Ben Johnston, Program Notes for his *Fourth String Quartet*

The Path to Scales and Tuning Systems

Imagine a melody that employs only octave-related pitches. How boring. Clearly, the ear perceives the octave to be a very large intervallic distance. Other, smaller intervals are obviously required if we are to achieve a greater degree of melodic expressiveness. Many cultures, including our own, have gone so far as to assume that pitches related by a 2:1 frequency ratio should be considered equivalent. This important musical principle, known as *octave equivalence,* lies at the very foundation of tuning theory. But exactly how does one go about dividing up the octave to create a scale? As it turns out, this is a rather complex question, so we will attempt here to present only the most basic principles. Our goal will be to provide you with some insight into the relationship that exists between the harmonic series and the seven-tone diatonic scale, perhaps the most important scale in Western music.

The word scale was derived from the Greek word, *skala*, meaning steps. The process of creating a scale may conveniently be thought of as dividing the octave (or other basic interval such as 3:1) into discrete steps. The size of each step is chosen to achieve a specific musical goal. Sometimes the goal is melodic in nature, other times it is harmonic, or perhaps it is some other musical goal established by a composer or instrument maker. The important thing to remember is that the size of the intervals that make up a given scale may be determined in a variety of ways. For example, the diatonic scale has been tuned in a variety of ways throughout history. During the Middle Ages (c476-1453) a Pythagorean tuning, a tuning derived from a series of six 3:2 fifths, was an accepted “standard” for tuning the seven tones of a diatonic scale. In the Renaissance (c1450-1600), however, a just tuning derived from the frequency relationships found in the first sixteen partials of the harmonic series emerged to rival Pythagorean tuning. Soon afterward a mean-tone temperamen*t* and a variety of other competing temperaments like Werkmeister and twelve-tone equal temperament were proposed. Each of these tuning systems offers its own set of benefits, as well as introduces new musical problems, so it is up to musicians to decide which system works best for a particular music.

J. Murray Barbor in his book *Tuning and Temperament: A Historical Survey* (1953), conveniently divides all tuning systems into two general categories:

1. **Tunings** - scale tones are derived from the harmonic series (i.e., whole-number ratios)
2. **Temperaments** - scale tones are derived by making adjustments to a tuning in order to achieve a desired musical result

Our currently accepted standard for tuning fixed-pitch instruments, *twelve-tone equal temperament* (12TET), emerged as a competitor in the early eighteenth century. (It should probably mentioned here that an equal division of the tone was suggested as early as the Greek philosopher/theorist Aristoxenus, b. c375-360 BCE.) Twelve-tone equal temperament was introduced as a solution to the problem that certain intervals did not remain the same size after modulation. Perhaps the best way to learn more about tuning theory is to do the math for yourself, performing the calculations necessary to derive a 7-tone diatonic and 12-tone chromatic scale in a given tuning system.

Click on a link to learn more about one of the following tuning systems:

1. Pythagorean tuning - a tuning based on the line of fifths
2. Just tuning - a tuning based on the intervals found in the lower part of the harmonic series
3. Mean-tone temperament - a temperament derived by making an adjustment to a just tuning
4. Twelve-tone equal temperament (12TET) - an accepted standard for tuning fixed-pitched instruments (such as the piano) since the time of J.S. Bach

> **On recognizable diatonic tunings**
> "The structure of recognizable diatonic tunings is basically an array of intricate interconnections...which are the very foundation of what is perceived as tonal harmonic motion, are shaped by the short-term span of human memory, the tolerance range of the human ear, and the peculiar manner in which intervals are perceived."
> Easley Blackwood, *The Structure of Recognizable Diatonic Tunings*

The Path to Consonance and Dissonance

The nineteenth-century physicist Herman von Helmholtz proposed a theory of consonance and dissonance based on the harmonic series, beating, and what he termed roughness. When two pure tones are very close in frequency (e.g., 440 and 442 Hz.) we perceive a regular variation in loudness, or amplitude, of the complex tone they combine to produce called beats. The phenomenon of beating is used by musicians every day in the tuning of their instruments. Though perhaps a gross oversimplification of the actual physical model, it is convenient to think of all musical tones as consisting of a series of harmonically-related pure tone components above the fundamental pitch we perceive. Additionally, recall that these pure tone components diminish significantly in loudness as one ascends the harmonic series. Furthermore, recent studies by music psychologists suggest that only the first six pure tone components contribute significantly to our classification of intervals as either consonant or dissonant (Campbell and Greated 1980). Though Helmholtz's theory is now considered to be too simplistic from the point of view of modern physics and music psychology, it is still worthy of our study because it provides a means of understanding the relationship between the harmonic series and the common "harmonious intervals". Since the time of Pythagoras, many cultures have noted that frequency ratios involving relatively small integers such as 1:1, 2:1, 3:2, 4:3, 5:3, 5:4, and 6:5 produce the common harmonious, stable or consonant intervals in music. Helmholtz proposed that the *degree of dissonance* produced by an interval is related to the degree of roughness produced by partials of the two fundamental tones that make up the interval. The term *roughness* was a quality Helmholtz associated with the number of semitone and whole-tone conflicts among the partials of the two fundamental tones. It should be stated, however, that the term roughness has an entirely different meaning in modern acoustics/psychoacoustics (See Beats, Critical Band, and Roughness for more information). Fig. 7 shows the partial conflicts for nine common intervals:

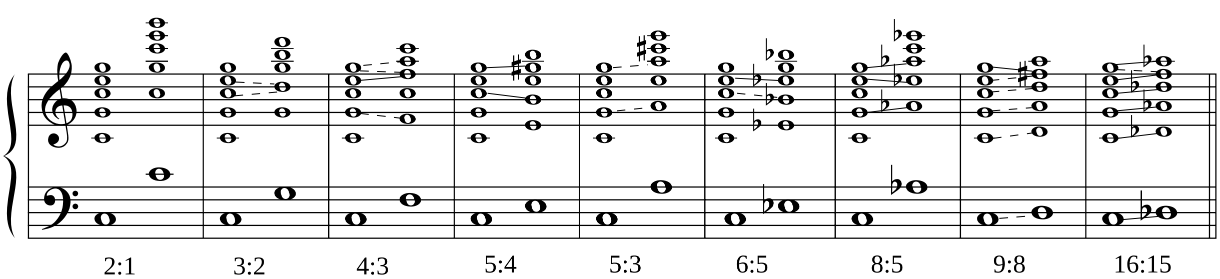

Fig. 7. Helmholtz degree of dissonance diagram after Campbell and Greated 1987.

Fundamental tones are notated in the bass clef staff. The first six partials of each fundamental tone are shown in the treble clef staff using smaller noteheads. Semitone conflicts between partials are indicated using a solid line. Whole-tone conflicts are indicated using a dotted line. It was assumed by Helmholtz that a semitone conflict was perceived to be more dissonant than a whole-tone conflict. For each interval shown in Fig. 7, compare the number of semitone and whole-tone conflicts that occur between partials. You will find that the interval 16:15 seems to produce the most roughness, followed by 9:8, 8:5, and so on. It should also be mentioned that modern music psychologists make a distinction between sensory consonance/dissonance and musical consonance/dissonance, and have also demonstrated that factors such as register and dynamic level play a large role in how we actually perceive the relative consonance or dissonance of musical intervals. Though there is much more to the principle of consonance and dissonance in music, Helmholtz's degree of dissonance theory nonetheless provides us with some insight into the harmonic series's role in our perception of musical intervals. Before we leave Fig. 7., use Helmholtz's degree of dissonance model to rank the nine intervals shown from most consonant to most dissonant. Did you come up with a different order than the order shown in Fig. 7?

The Path to Timbre

In the 1970's John Grey, James A. Moorer, and J. C. Risset were some of the first musicians/scientists to use computers to analyze tones produced by acoustic instruments. Using computers they were able to isolate the individual partials of a variety of acoustic instrument tones and show how each partial's amplitude progressed independently through time. Fig. 8 displays an analysis of a trumpet tone produced by Grey and Moorer.

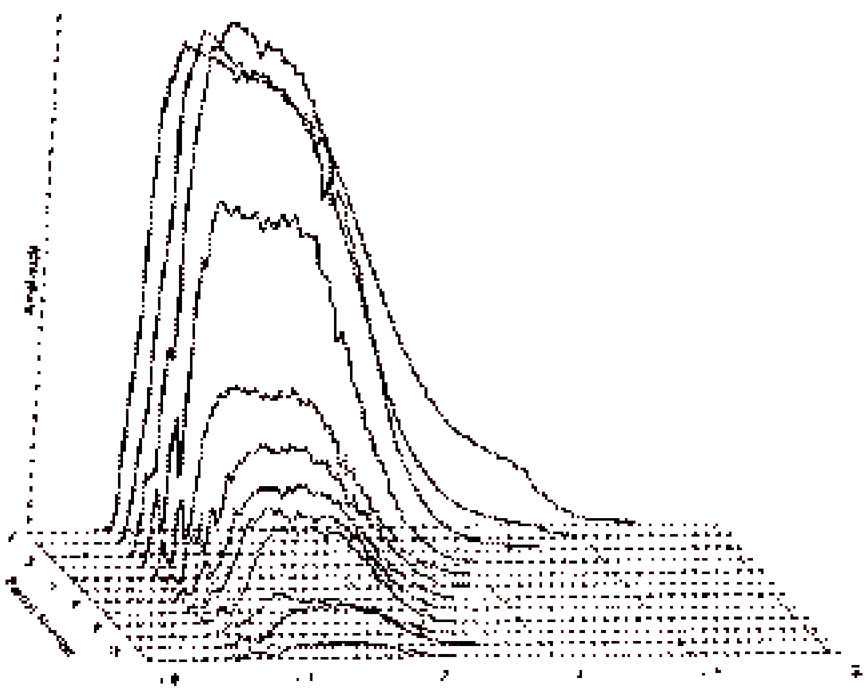

Fig. 8. The amplitude progressions of the partials of a trumpet tone.

The graph in Fig. 8 is 3-dimensional. The three dimensions are: 1) time, read left-to-right, 2) amplitude, read down-to-up, and 3) partial number, read back-to-front. This type of diagram is called a spectral diagram because it unveils the frequency components of the acoustical sound source. Recall that when light is passed through a prism, the individual frequency component (colors) of white light are unveiled because they travel at different speeds through the glass medium. Similarly, Fig. 8 shows the first twelve partials of a trumpet tone. The first .5 seconds of the trumpet tone are displayed. Notice how the lower partials rise first. Also notice how the lowest tone components are both the loudest and the last to decay. From this analysis we can see that the basic building blocks of a trumpet tone can be expressed as harmonic partials (or perhaps more accurately, nearly-harmonic partials). In 1822, the French mathematician Jean Baptist Fourier (1768-1830) came forth with a mathematical proof demonstrating that any waveform or signal, no matter how complex, could be reduced to an infinite set of sine wave components. This type of analysis is called Fourier analysis. Most musical instruments produce waveforms that have been found to be *periodic* or *quasi-periodic*, meaning that a basic waveshape is repeated over-and-over throughout the course of the tone. Fourier analysis can be implemented on modern personal computers using a computationally intensive algorithm known as the *Fast-Fourier Transform* (FFT). FFT algorithms can run on today's fast personal computers in real-time, opening up myriad possibilities for application in the music industry as demonstrated by FFT algorithm's use in MP3 and other compression techniques.

On the human auditory system
"…*periodic* is a term frequently used by scientists and engineers to describe a large class of signals whose component parts fall in the harmonic series, but in fact signals in nature are not periodic, and the human auditory system knows this to be so."
John Chowning, *Perceptual Fusion and Auditory Perspective*

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